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Simultaneous Rational Approximation of a Function and Its Derivatives in the Complex Plane

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Some aspects of simultaneous rational approximation of a function $f(z)$ and its derivatives on the unit circle are investigated. The function $f(z)$ is assumed to be analytic in some annulus containing the unit circle, and given a nonnegative integer l,

$$\|f\| = \max\{\|f^{(j)}\|_{p,1} : 0 \leqslant j \leqslant l\},$$

where $\|\cdot\|_{p,1}$ is the usual L_p norm $(1 \leqslant p \leqslant \infty)$ on the unit circle. It is shown that the polynomial of simultaneous best approximation in the above norm, is just a polynomial of best approximation to $f^{(l)}$, suitably integrated. Further, sharp asymptotic results are obtained for the case where the order of the derivative, namely l, tends to infinity. For example, if f is meromorphic in $\mathbb{C}$ of finite order ρ, with v poles in all, none lying on the unit circle, and if $0 \leqslant \mu < \min\{1, 1/\rho\}$, then

$$\limsup_{m \to \infty} \left(\min_{R \in \mathcal{R}_{mv}} \max_{0 \leqslant j \leqslant \mu m} \|f^{(j)} - R^{(j)}\|_{p,1} \right)^{1/m \log m} = e^{\mu - 1/\rho}.$$

1. INTRODUCTION

Although quite a lot of attention has been given to simultaneous rational or polynomial approximation of a function and its derivatives on a real interval [7–9], not much has been written about this problem in the complex domain. Furthermore, as far as the authors can determine, no one has investigated (for either real intervals or complex domains) the asymptotic

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behaviour of the error when the order of the derivative tends to infinity at the same time as the degree of the approximating rational (or polynomial) functions tends to infinity.

In Section 3, we assume that f is meromorphic of finite order ρ, and using elementary methods, we establish results on the asymptotic behaviour of the error. Theorem 3.2 deals with the case when f may have infinitely many poles, while in Theorem 3.3, a result which is more general in one direction, is obtained for the case where f has poles of finite total multiplicity.

In Section 4, we obtain converse results expressing the order of a meromorphic function with finitely many poles in terms of the asymptotic behaviour of the error. Theorem 4.2 deals with the case of approximation by polynomials. The more difficult case of approximation by rational functions with denominator of fixed positive degree, is treated in our main result, Theorem 4.5. The forward and converse assertions of Sections 3 and 4 are summarized in Corollaries 4.3 and 4.6.

It is well known that the partial sums of the Maclaurin series of an entire function f are asymptotically the best possible polynomial approximations to f on a circle. The results in this note show that similar statements are true for simultaneous rational approximations to a meromorphic function. More precisely, if $f = g/h$, where g is entire of finite order, and h is a polynomial, then partial sums of g divided by h yield asymptotically sharp rational approximations of fixed denominator degree, to f. In that sense, this paper presents no surprises.

In Section 5, two extensions of the results in Sections 3 and 4 are mentioned without proof. Finally, in Section 6, we investigate the role of the last derivative and show that the polynomial of best approximation of f in the norm

$$\|f\| = \max\{\|f^{(j)}\|_{p,1} : 0 \leqslant j \leqslant l\},$$

is just a polynomial of best approximation to $f^{(l)}$, suitably integrated. This is the complex analogue of a result of Meir and Sharma [8].

2. Notation

Let $r > 0$. We denote by $\mathscr{A}_r$ the class of functions analytic in some annulus $\{z : r - \varepsilon < |z| < r\}$, where $\varepsilon > 0$ may depend on the function. For each $f \in \mathscr{A}_r$ and $1 \leqslant p \leqslant \infty$, we let

$$\|f\|_{p,r} = \begin{cases} \lim\limits_{s \to r-} \left\{\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(se^{i\theta})|^p \, d\theta\right\}^{1/p}, & 1 \leqslant p < \infty \\ \lim\limits_{s \to r-} \left\{\max\limits_{|z|=s} |f(z)|\right\}, & p = \infty, \end{cases}$$

whenever these norms are defined. Further for all nonnegative real numbers l, for all $r>0$ and $1 \leqslant p \leqslant \infty$, we set

$$\|f\|_{p,r,l} = \max_{0 \leqslant j \leqslant l} \|f^{(j)}\|_{p,r}$$

whenever these norms are defined.

For nonnegative integers m and n, $\mathscr{P}_m$ denotes the class of polynomials of degree at most m, and $\mathscr{R}_{mn}$ denotes the class of rational functions of type (m, n) (i.e., rational functions with numerator of degree at most m and denominator of degree at most n). If $f \in \mathscr{A}_1$ and $\|f\|_{p,1,l} < \infty$, we let

$$e_f(l; m; n; p) = \min_{R \in \mathscr{R}_{mn}} \|f - R\|_{p,1,l}, \tag{2.1}$$

so that $e_f(l; m; n; p)$ is the error in best approximation of f by rational functions of type (m, n) in the norm $\|\cdot\|_{p,1,l}$. It is easy to see that the minimum in (2.1) is actually attained by some $R \in \mathscr{R}_{mn}$, but it is known that R need not be unique. Finally, $[x]$ denotes the largest integer $\leqslant x$.

3. Asymptotic Results

In this section the asymptotic behaviour of $e_f(l; m; n; p)$ is investigated when f is meromorphic of finite order. The case where l is fixed is very similar to the case $l = 0$ and so we study the more interesting case where l approaches infinity as m, and possibly n, approach infinity. For definiteness, we let $l = \mu m$, where μ is a fixed nonnegative number.

Lemma 3.1. *Let $0 < \varepsilon < 1$. Let $f(z) = g(z)/h(z)$, where both g, h are analytic in $\{z: |z| \leqslant 1 + 2\varepsilon\}$. Further let h have no zeroes in the annulus $A = \{z: 1 - 2\varepsilon \leqslant |z| \leqslant 1 + 2\varepsilon\}$. Let $R = P/Q$, where P, Q are polynomials and Q has no zeroes in A. Then for any positive integer l,*

$$\|f - R\|_{p,1,l} \leqslant K_1 l! \, \varepsilon^{-l} \frac{\max\{\|h - Q\|_{\infty,1+\varepsilon}, \|g - P\|_{\infty,1+\varepsilon}\}}{\min\{|Q(t)|: |t| = 1 \pm \varepsilon\}}, \tag{3.1}$$

where $K_1 = 1 + \max\{\|f\|_{\infty,1+\varepsilon}, \|f\|_{\infty,1-\varepsilon}\}$.

Proof. Fix z such that $|z| = 1$. Consider the circle C, centre z, radius ε. As $f - R$ is analytic inside and on C, for $j \geqslant 0$,

$$\begin{aligned} |(f-R)^{(j)}(z)| &= \left| \frac{j!}{2\pi i} \int_C \frac{(f-R)(t)}{(t-z)^{j+1}} \, dt \right| \\ &\leqslant j! \, \varepsilon^{-j} \max\{|(f-R)(t)|: |t| = 1 \pm \varepsilon\}, \end{aligned} \tag{3.2}$$

by the maximum modulus principle. Now for $|t| = 1 \pm \varepsilon$,

$$\begin{aligned} |(f-R)(t)| &= |g(t)(Q-h)(t) + h(t)(g-P)(t)|/|h(t)\,Q(t)| \\ &\leqslant |Q(t)|^{-1}\,\{|f(t)| \cdot |(Q-h)(t)| + |(g-P)(t)|\} \\ &\leqslant K_1 \max\{\|Q-h\|_{\infty,1+\varepsilon}, \|g-P\|_{\infty,1+\varepsilon}\} \\ &\quad \div \min\{|Q(t)|: |t| = 1 \pm \varepsilon\}, \end{aligned} \tag{3.3}$$

by analyticity of g, h. Now (3.1) follows from (3.2) and (3.3). ∎

As a consequence of this lemma, we can prove a result for meromorphic functions, which is related (for $\mu = 0$) to Theorem 3 in Karlsson [5].

THEOREM 3.2. *Let f be meromorphic in $\mathbb{C}$ with no poles on the unit circle. Further let f have order at most $\rho < \infty$ and let $0 \leqslant \mu < \infty$, $1 \leqslant p \leqslant \infty$. Then*

$$\limsup_{m \to \infty} e_f(\mu m; m; m; p)^{1/m \log m} \leqslant e^{\mu - 1/\rho}. \tag{3.4}$$

(If $\rho = 0$, $e^{-1/\rho}$ is taken as 0.)

Proof. Using elementary theory of meromorphic functions [4], we can write $f = g/h$, where g, h are entire and have at most order ρ. Further as f is analytic on the unit circle, we can assume there exists $0 < \varepsilon < 1$ such that h does not vanish in th annulus $A = \{z: 1 - 2\varepsilon \leqslant |z| \leqslant 1 + 2\varepsilon\}$.

Let P_m, Q_m be the $(m+1)$th partial sums of the Maclaurin series of g, h, respectively. Then for large m, Q_m has no zeroes in A and

$$\lim_{m \to \infty} (\min\{|Q_m(t)|: |t| = 1 \pm \varepsilon\}) = \min\{|h(t)|: |t| = 1 \pm \varepsilon\} > 0. \tag{3.5}$$

If $l = l(m) = [\mu m]$, Stirling's formula shows that

$$\lim_{m \to \infty} l!^{1/m \log m} = e^{\mu}. \tag{3.6}$$

Finally it follows trivially from the rate of convergence to 0 of the Maclaurin series coefficients of entire functions of order $\leqslant \rho$ [6, Theorem 2], that

$$\begin{aligned} &\limsup_{m \to \infty} \|g - P_m\|_{\infty,1+\varepsilon}^{1/m \log m} \leqslant e^{-1/\rho}, \\ &\limsup_{m \to \infty} \|h - Q_m\|_{\infty,1+\varepsilon}^{1/m \log m} \leqslant e^{-1/\rho}. \end{aligned} \tag{3.7}$$

Now (3.4) follows from (3.1), (3.5), (3.6) and (3.7). ∎

It seems likely that one can replace the right-hand side of (3.4) by $\max\{e^{1-1/\rho}, e^{\mu(1-1/\rho)}\}$ if $\mu > 1$. We are able to prove this sharpened estimate under the additional restriction that f has poles of finite total multiplicity. Although we formulate Theorem 3.3 for approximation by elements of $\mathscr{R}_{m\nu}$, one can replace ν in the left-hand side of (3.8) by any sequence of integers no smaller than ν.

THEOREM 3.3. *Let f be meromorphic in $\mathbb{C}$ with poles of total multiplicity $\nu<\infty$, none lying on the unit circle. Further, let f have order at most $\rho<\infty$, and let $0\leqslant\mu<\infty$ and $1\leqslant p\leqslant\infty$. Then*

$$\limsup_{m\to\infty} e_f(\mu m; m; \nu; p)^{1/m\log m} \leqslant \begin{cases} e^{\mu-1/\rho}, & \mu\leqslant 1 \\ \max\{e^{1-1/\rho}, e^{\mu(1-1/\rho)}\}, & \mu>1. \end{cases} \tag{3.8}$$

Proof. If $\mu\leqslant 1$, the proof is very similar to that of Theorem 3.2: Write $f=g/h$, where g is entire of order at most $\rho<\infty$ and h is a polynomial of degree ν having no zeroes on the unit circle. We then choose P_m, Q_m as in Theorem 3.2 and obtain (3.8) for $\mu\leqslant 1$.

Now suppose $\mu>1$. We can write $f=f^*+R^*$, where f^* is entire of order at most ρ and R^* (the principal part of f) is a rational function of type $(\nu-1, \nu)$. Now if P is any polynomial of degree $\leqslant m-\nu$, then $P+R^*$ is a rational function of type (m, ν). Thus

$$\begin{aligned} e_f(\mu m; m; \nu; p) &\leqslant \min_{P\in\mathscr{P}_{m-\nu}} \|(f^*+R^*)-(P+R^*)\|_{p,1,\mu m} \\ &\leqslant \max\{e_{f^*}(m-\nu; m-\nu; 0; p), \max_{m-\nu+1\leqslant j\leqslant\mu m} \|f^{*(j)}\|_{p,1}\}, \end{aligned} \tag{3.9}$$

since $P^{(j)}\equiv 0$ if $j>m-\nu$, $P\in\mathscr{P}_{m-\nu}$. From what we already know for the case $\mu\leqslant 1$, and as ν is fixed,

$$\limsup_{m\to\infty} e_{f^*}(m-\nu; m-\nu; 0; p)^{1/m\log m}\leqslant e^{1-1/\rho}. \tag{3.10}$$

Next, let $m-\nu+1\leqslant j\leqslant\mu m$ and apply Lemma 3.1 to f^* (with $g=f^*-P$, $h=1$, $\varepsilon=1$, $Q=1$). We obtain

$$\begin{aligned} \|f^{*(j)}\|_{p,1} &= \min\{\|(f^*-P)^{(j)}\|_{p,1} : P\in\mathscr{P}_{j-1}\} \\ &\leqslant K_1\, j!\, \min\{\|f^*-P\|_{\infty,2} : P\in\mathscr{P}_{j-1}\}, \end{aligned} \tag{3.11}$$

where K_1 depends only on f^*. Using (3.11), Stirling's formula, and the fact that f^* has order $\leqslant\rho$, we see

$$\begin{aligned} &\limsup_{m\to\infty} \Big(\max_{m-\nu+1\leqslant j\leqslant\mu m} \|f^{*(j)}\|_{p,1}\Big)^{1/m\log m} \\ &\quad\leqslant \limsup_{m\to\infty} \Big(\max_{m-\nu+1\leqslant j\leqslant\mu m} e^{(j\log j)(1-1/\rho)}\Big)^{1/m\log m} \\ &\quad= \max\{e^{1-1/\rho}, e^{\mu(1-1/\rho)}\}. \end{aligned} \tag{3.12}$$

Now (3.9), (3.10), and (3.12) yield the result. ∎

The difference between the cases $\mu \leqslant 1$ and $\mu > 1$ evidently arises from the fact that for $\mu > 1$, the order of the derivative (namely $[\mu m]$) exceeds the order of the rational or polynomial approximation.

4. Converse Results

As a first step towards establishing a converse result for Theorem 3.3, we prove

Lemma 4.1. *Let $f(z) = \sum_{j=0}^{\infty} f_j z^j$ be analytic in $|z| < 1$. Let $1 \leqslant p \leqslant \infty$. Further let Λ be an infinite sequence of positive integers. Assume there exist finite θ and finite nonnegative μ such that*

$$\limsup_{\substack{m \to \infty \\ m \in \Lambda}} e_f(\mu m; m; 0; p)^{1/m \log m} \leqslant e^{\theta}. \tag{4.1}$$

Let ρ be the number uniquely defined by the following equations:

$$\begin{aligned} 1/\rho &= \mu - \theta && \text{if} \quad \mu \leqslant 1, \\ &= 1 - \theta && \text{if} \quad \mu > 1 \text{ and } \theta < 0, \\ &= 1 - \theta/\mu && \text{if} \quad \mu > 1 \text{ and } \theta \geqslant 0. \end{aligned} \tag{4.2}$$

Then if $0 \leqslant \mu \leqslant 1$ or if $\theta < 0$,

$$\limsup_{\substack{m \to \infty \\ m \in \Lambda}} |f_{m+1}|^{1/m \log m} \leqslant e^{-1/\rho}, \tag{4.3A}$$

while if $\mu > 1$ and $\theta \geqslant 0$,

$$\limsup_{\substack{m \to \infty \\ m \in \Lambda}} (\max\{|f_j|^{1/j \log j} : \mu m + 1 \leqslant j \leqslant \mu(m+1)\}) \leqslant e^{-1/\rho}. \tag{4.3B}$$

Proof. We prove first (4.3B). Let $l = [\mu m]$. Then if $\mu m + 1 \leqslant j \leqslant \mu(m+1)$ and $P \in \mathscr{P}_m$, we have for $r < 1$,

$$\begin{aligned} f_j = (f - P)^{(j)}(0)/j! &= [(f - P)^{(l)}]^{(j-l)}(0)/j! \\ &= \frac{(j-l)!}{j!} \frac{1}{2\pi i} \int_{|z| = r} (f - P)^{(l)}(z)\, z^{-(j-l+1)}\, dz. \end{aligned}$$

As P was any polynomial of degree $\leqslant m$ and as the L_p norm increases monotonically with r, we deduce

$$|f_j| \leqslant \{(j-l)!/j!\}\, e_f(\mu m; m; 0; p). \tag{4.4}$$

Now $\lim_{m\to\infty} j/m=\mu$ uniformly for the range of j considered while $\lim_{m\to\infty} l/m=\mu$. Using Stirling's formula, we deduce

$$\limsup_{m\to\infty}\left(\max\{((j-l)!/j!)^{1/j\log j}:\mu m+1\leqslant j\leqslant \mu(m+1)\}\right)=e^{-1}. \tag{4.5}$$

Further from (4.1), (4.2), we see that

$$\begin{aligned}&\limsup_{\substack{m\to\infty\\ m\in\Lambda}}\left(\max\{e_f(\mu m;m;0;p)^{1/j\log j}:\mu m+1\leqslant j\leqslant\mu(m+1)\}\right)\\ &\quad\leqslant e^{\theta/\mu}=e^{1-1/\rho}.\end{aligned} \tag{4.6}$$

Then (4.3B) follows from (4.4)–(4.6).

The cases where $0\leqslant\mu\leqslant 1$ or $\theta<0$ are easier: Let $\mu^-=\min\{\mu,1\}$ and $l=[\mu^-m]$. As before one obtains

$$|f_{m+1}|\leqslant\{(m+1-l)!/(m+1)!\}\,e_f(\mu m;m;0;p),$$

and using (4.1) and (4.2),

$$\begin{aligned}\limsup_{m\to\infty}|f_{m+1}|^{1/m\log m}&\leqslant e^{\theta}\limsup_{m\to\infty}\{(m+1-l)!/(m+1)!\}^{1/m\log m}\\ &=e^{\theta-\mu^-}=e^{-1/\rho}. \quad\blacksquare\end{aligned}$$

We can now prove the converse of Theorem 3.3 for the case $\nu=0$, in which there is no restriction on the size of μ.

THEOREM 4.2. *Let $f(z)$ be analytic in $|z|<1$. Let $1\leqslant p\leqslant\infty$. Assume there exist finite θ and finite nonnegative μ such that*

$$\limsup_{m\to\infty} e_f(\mu m;m;0;p)^{1/m\log m}\leqslant e^{\theta}.$$

Let ρ, the number uniquely defined by the equations (4.2), *be positive. Then f is the restriction to $\{z:|z|<1\}$ of an entire function of order at most ρ.*

Proof. Write $f(z)=\sum_{j=0}^{\infty}f_jz^j$. By Theorem 2 in [6], it suffices to show

$$\limsup_{j\to\infty}|f_j|^{1/j\log j}\leqslant e^{-1/\rho}. \tag{4.7}$$

Now (4.1) holds for Λ being the sequence of all positive integers. In the cases where $\mu\leqslant 1$ or $\theta<0$, (4.7) follows from (4.3A). In the case where $\mu>1$ and $\theta\geqslant 0$, (4.7) still follows from (4.3B), since the set of all positive integers exceeding μ is contained in the set

$$\bigcup_{m=1}^{\infty}\{j:\mu m+1\leqslant j\leqslant\mu(m+1)\}. \quad\blacksquare$$

We note that ρ, defined by (4.2), is positive provided $\theta < \mu$, and that this latter condition is satisfied in Theorem 3.3 (with the obvious choices of θ). As an immediate corollary of Theorems 3.3 and 4.2 we have

COROLLARY 4.3. *If $f(z)$ is entire of order $\rho < \infty$ and if $1 \leqslant p < \infty$, we have*

$$\limsup_{m \to \infty} e_f(\mu m; m; 0; p)^{1/m \log m} = \begin{cases} e^{\mu - 1/\rho}, & \mu \leqslant 1 \\ \max\{e^{1-1/\rho}, e^{\mu(1-1/\rho)}\}, & \mu > 1. \end{cases}$$

The converse of Theorem 3.3 for $v > 0$ seems much harder, and we can prove it only for μ, ρ such that $\theta < 0$ in (4.2). The techniques used below are largely due to Cirka [1], Grigorjan [3] and both their work owes much to Gončar. First, we note the following lemma from [1, p. 126]:

LEMMA 4.4. *Let $Q(z)$ be a polynomial of degree $\leqslant m$ normalized so that $\|Q\|_{\infty,1} = 1$. Then the set $\{z: |z| < \varepsilon^{-1/3}, |Q(z)| < \varepsilon^m\}$ can be covered by a finite number of balls, the sum of whose diameters does not exceed $A\varepsilon^{1/3}$, where A is an absolute constant.*

THEOREM 4.5. *Let $f \in A_1$. Assume there exist $1 \leqslant p \leqslant \infty, 0 \leqslant \mu < \infty$, $\theta < 0$ and a nonnegative integer v such that*

$$\limsup_{m \to \infty} e_f(\mu m; m; v; p)^{1/m \log m} \leqslant e^{\theta}. \tag{4.8}$$

Let ρ be the number uniquely defined by Eqs. (4.2). Then f is the restriction of a function meromorphic in $\mathbb{C}$ of order at most $\rho < \infty$ and with poles of total multiplicity $\leqslant v < \infty$.

Proof. We prove this in steps. Let $R \in \mathscr{R}_{mv}$ satisfy $\|f - R_m\|_{p,1,\mu m} = e_f(\mu m; m; v; p)$, $m = 1, 2, \ldots$.

Step 1. Let $\Lambda = \{m_k\}_{k=1}^{\infty}$ be an increasing sequence of integers such that

$$2^k \leqslant m_k \leqslant 2^{k+1}, \qquad k = 1, 2, 3, \ldots. \tag{4.9}$$

We show that as $m \to \infty, m \in \Lambda$, $R_m^{(j)}$ converges outside a thin set for $j = 0, 1, 2, \ldots$. Write $g_k = R_{m_{k+1}} - R_{m_k} = P_{k,0}/Q_k$, a rational function of type $(m_{k+1} + v, 2v)$. We normalize Q_k so that $\|Q_k\|_{\infty,1} = 1$. By induction on j, we see that

$$g_k^{(j)}(z) = P_{k,j}(z)/(Q_k(z))^{j+1}, \tag{4.10}$$

where $P_{k,j}$ is a polynomial of degree at most $m_{k+1}+\nu+(2\nu-1)\,j$. Then for $0\leqslant j\leqslant \mu m_k$, (4.9) shows that $P_{k,j}$ has degree at most $A_1 m_k$, where A_1 is independent of j and k.

Next, if $0<\eta<1$ is fixed, Cauchy's integral formula and (4.10) show that

$$\begin{aligned}\|P_{k,j}\|_{\infty,1-\eta}&\leqslant \eta^{-1}\|P_{k,j}\|_{p,1}\leqslant \eta^{-1}\|g_k^{(j)}\|_{p,1}\\&=\eta^{-1}\|(f-R_{m_k})^{(j)}+(R_{m_{k+1}}-f)^{(j)}\|_{p,1}.\end{aligned}\tag{4.11}$$

Let $\varepsilon_k=\exp(-\log m_k/\log\log m_k)$ and $r_k=\log m_k$, $k=1,2,\ldots$. Further let $\mathscr{E}_k=\{z\colon |z|<\varepsilon_k^{-1/3},\ |Q_k(z)|<\varepsilon_k^{2\nu}\}$. Then for all $0\leqslant j\leqslant \mu m_k$, all $|z|\leqslant r_k$, $z\notin\mathscr{E}_k$, we have by (4.8), (4.10), (4.11), and the Walsh–Bernstein lemma [11, p. 77],

$$\begin{aligned}|g_k^{(j)}(z)|&\leqslant \varepsilon_k^{-2\nu(j+1)}\|P_{k,j}\|_{\infty,r_k}\\&\leqslant \varepsilon_k^{-2\nu(\mu m_k+1)}(r_k/(1-\eta))^{A_1 m_k}\|P_{k,j}\|_{\infty,1-\eta}\\&\leqslant e^{m_k\log m_k(\theta+o(1))}.\end{aligned}\tag{4.12}$$

Here, of course, the $o(1)$ term is independent of z and j.

Now, by Lemma 4.4, $\mathscr{E}_k$ can be covered by open balls, the sum of whose diameters does not exceed $A\varepsilon_k^{1/3}$. Let $\mathscr{F}_k=\bigcup_{j\geqslant k}\mathscr{E}_j$. Then $\mathscr{F}_k$ is covered by balls, the sum of whose diameters is at most $A\sum_{j\geqslant k}\varepsilon_j^{1/3}\to 0$ as $k\to\infty$ (by (4.9) and our choice of ε_k). Since $\theta<0$, we deduce from (4.12) that for $|z|\leqslant r_k$, $z\notin\mathscr{F}_k$,

$$\left|\sum_{l\geqslant k} g_l^{(j)}(z)\right|\leqslant e^{m_k\log m_k(\theta+o(1))}$$

and so

$$\lim_{\substack{m\to\infty\\ m\in\Lambda}} R_m^{(j)}(z)=f_j(z)\qquad\text{(say)}$$

exists. Further, if $m=m_k$, we have uniformly for $|z|\leqslant r_k$ such that $z\notin\mathscr{F}_k$,

$$\begin{aligned}\max_{0\leqslant j\leqslant\mu m}|f_j(z)-R_m^{(j)}(z)|&=\max_{0\leqslant j\leqslant \mu m}\left|\sum_{l\geqslant k} g_l^{(j)}(z)\right|\\&\leqslant e^{m\log m(\theta+o(1))}.\end{aligned}\tag{4.13}$$

Step 2. Choose a subsequence Λ_1 of Λ such that as $m\to\infty$, $m\in\Lambda_1$, the poles of R_m converge to at most ν points $\alpha_1,\alpha_2,\ldots,\alpha_\nu$, say. We show that f may be continued analytically to $\mathbb{C}\backslash\{\alpha_1,\alpha_2,\ldots,\alpha_\nu\}$, and that $R_m\to f$ in this latter set, as $m\to\infty$, $m\in\Lambda_1$.

Let $0<\delta<1$. For large $m'=m_{k'}\in\Lambda_1$, $\mathscr{F}_{k'}$ can be covered by balls whose sum of diameters is less than δ. Hence one can find circles centered on

$\alpha_1, \alpha_2, \ldots, \alpha_\nu$ of radius between δ and 2δ, and a large circle C, centre 0, with radius between $1/\delta - \delta$ and $1/\delta$ not intersecting $\mathscr{F}_{k'}$. We can assume δ is so small that none of these circles intersect. Let $\mathscr{D}$ denote the bounded connected open set whose boundary $\delta\mathscr{D}$ is formed by these circles. Since $\mathscr{F}_k \subset \mathscr{F}_{k'}$ for $k \geqslant k'$, we see $\delta\mathscr{D}$ does not intersect $\mathscr{F}_k$ for $k \geqslant k'$.

Furthermore, for large $m \in \Lambda_1$, R_m is analytic inside $\mathscr{D}$ and on $\delta\mathscr{D}$, and by (4.13), $\{R_m^{(j)}\}_{m \in \Lambda_1}$ is uniformly bounded on $\delta\mathscr{D}$ for each $j \geqslant 0$, and converges on $\delta\mathscr{D}$. By the convergence-continuation theorems (Stieltjes–Vitali theorem), $R_m^{(j)}$ converges uniformly in $\mathscr{D}$ to a function f_j analytic in $\mathscr{D}$ as $m \to \infty, m \in \Lambda_1$. Furthermore as $\mathscr{D}$ has nonempty intersection with the open annulus in which f is analytic, we see from (4.8) and (4.13) and well-known uniqueness results in H_p spaces of finitely connected domains that f_0 continues f analytically to $\mathscr{D}$. The uniform convergence of $\{R_m^{(j)}\}_{m \in \Lambda_1}$ to f_j ensures that $f_j = f_0^{(j)} = f^{(j)}$.

Finally, as $\delta > 0$ was arbitrary, it follows that f can be continued to a function analytic in $\mathbb{C} \backslash \{\alpha_1, \alpha_2, \ldots, \alpha_\nu\}$. It is easy to see that f can have at most poles of total multiplicity $\nu < \infty$. For if S_m is the denominator of R_m, $m \in \Lambda_1$, normalized to be a monic polynomial of degree at most ν, we know

$$\lim_{\substack{m \to \infty \\ m \in \Lambda_1}} R_m(z)\, S_m(z) = f(z) \prod_{k=1}^{\nu} (z - \alpha_k) \tag{4.14}$$

uniformly in compact subsets of $\mathbb{C} \backslash \{\alpha_1, \alpha_2, \ldots, \alpha_\nu\}$. Since each $R_m S_m$ is a polynomial, these functions must converge as $m \to \infty, m \in \Lambda_1$ for $z = \alpha_1, \alpha_2, \ldots, \alpha_\nu$ also, and so the right-hand side of (4.14) is entire. Hence f has poles of total multiplicity at most ν.

Step 3. We show f has order at most ρ. Write $f = f^* + \mathscr{P}$ and $R_m = \mathscr{P}_m^* + \mathscr{P}_m$, where $\mathscr{P}$, $\mathscr{P}_m^*$ are, respectively, the principal parts of f and R_m in $\mathbb{C}$. It is easy to see then that $\mathscr{P}^{(j)}$, $\mathscr{P}_m^{(j)}$ are the principal parts of $f^{(j)}$, $R_m^{(j)}$, respectively, in $\mathbb{C}$. Furthermore we see f^* is entire and $\mathscr{P}_m^*$ is a polynomial of degree at most m.

Let Λ_1, δ, C, and $\mathscr{D}$ be as in Step 2. Recall that C was a circle with centre 0 of diameter between $1/\delta$ and $1/\delta - \delta$. By (4.13) and as $\mathscr{F}_k$ did not intersect C for $k \geqslant k'$, we have for $m = m_k$,

$$\max_{z \in C} \max_{0 \leqslant j \leqslant \mu m} |f^{(j)}(z) - R_m^{(j)}(z)| \leqslant e^{m \log m(\theta + o(1))}.$$

By Theorem 1 in Grigorjan [3], there exists K depending only on C such that for $m = m_k$,

$$\max_{z \in C} \max_{0 \leqslant j \leqslant \mu m} |f^{*(j)}(z) - P_m^{*(j)}(z)| \leqslant K(2\nu)(\mu m + 1)\, e^{m \log m(\theta + o(1))}.$$

As C contains the unit circle in its interior, we deduce

$$\limsup_{\substack{m \to \infty \\ m \in \Lambda_1}} e_{f^*}(\mu m; m; 0; \infty)^{1/m \log m} \leqslant e^{\theta}. \tag{4.15}$$

Since Λ_1 was any subsequence of Λ for which the poles of R_m, $m \in \Lambda_1$, converged and since f^* is independent of Λ_1 (being the unique entire part of the unique analytic continuation of f), we deduce that (4.15) holds with Λ_1 replaced by Λ. Finally, because Λ was any sequence for which (4.9) held, we claim that (4.15) holds with Λ_1 replaced by the sequence of all positive integers. For suppose we can find an increasing sequence Λ_0 for which

$$\limsup_{\substack{m \to \infty \\ m \in \Lambda_0}} e_{f^*}(\mu m; m; 0; \infty)^{1/m \log m} > e^{\theta}. \tag{4.16}$$

We can assume $\Lambda_0 = \{m_k\}_1^{\infty}$, where $m_k \geqslant 2^k$, $k = 1, 2, \ldots$. By "filling in" gaps in the sequence Λ_0 in an obvious manner, we can ensure that (4.9) holds for Λ_0, while (4.16) still holds for the enlarged sequence. This contradicts (4.15), which holds with Λ_1 replaced by Λ_0.

Thus (4.15) holds for the full sequence of positive integers replacing Λ_1. By Theorem 4.2, f^* has order at most ρ. Hence f, being the sum of f^* and a rational function, has order at most ρ. Note that $\rho > 0$ as $\theta < 0$. ∎

For $\mu = 0$, Theorem 4.5 is related to Theorem 2 in Saff [10].

COROLLARY 4.6. *Let f be meromorphic in $\mathbb{C}$ of order $\rho < \infty$ with poles of total multiplicity $\leqslant \nu < \infty$, none lying on the unit circle. If either $\mu \leqslant 1$ and $\mu < 1/\rho$, or $\mu > 1$ and $1 - 1/\rho < 0$, then*

$$\limsup_{m \to \infty} e_f(\mu m; m; \nu; p)^{1/m \log m} = \begin{cases} e^{\mu - 1/\rho}, & \mu \leqslant 1 \\ e^{1 - 1/\rho}, & \mu > 1 \end{cases} \tag{4.17}$$

for all $1 \leqslant p \leqslant \infty$.

Remarks. (a) It seems certain that (4.17) should hold without the restrictions on μ and ρ above, provided one replaces $e^{1-1/\rho}$ by $\max\{e^{1-1/\rho}, e^{\mu(1-1/\rho)}\}$.

(b) Even for $\mu = 0$, (4.17) does not seem to appear in the literature.

There is no possibility of a similar converse result for diagonal rational approximations. For if we let $\varepsilon > 0$ and $f(z) = \sum_{n=1}^{\infty} n^{-(n^2)}/(z - \alpha_n)$, where the $\{\alpha_n\}$ are dense in $\mathbb{C} \backslash \{z\colon 1 > |z| > 1 - \varepsilon\}$ and $|1 - |\alpha_n|| \geqslant 1/n$, $n = 1, 2, \ldots$, then f is analytic only in $\{z\colon 1 > |z| > 1 - \varepsilon\}$, but

$$\lim_{m \to \infty} \Big\| f(z) - \sum_{n=1}^{m} n^{-(n^2)}/(z - \alpha_n) \Big\|_{\infty, 1, \mu m}^{1/m \log m} = 0.$$

Thus (3.4) holds with $\rho = 0$ for all $\mu \geqslant 0$. So (3.4) does not entail that f can be continued analytically to a meromorphic or quasi-meromorphic function outside its annulus of analyticity.

Even for entire functions, a converse result is not possible. As Karlsson [5, p. 42] remarks, for $f(z) = e^z$, we have

$$\lim_{m \to \infty} e_f(0; m; m; \infty)^{1/m \log m} = e^{-2} = e^{-2/\rho}$$

but one can construct entire functions f of order ρ for which

$$\limsup_{m \to \infty} e_f(0; m; m; \infty)^{1/m \log m} = e^{-1/\rho}.$$

Still, we offer the following conjecture which will be of interest even for $\mu = 0$:

Conjecture. Let f be meromorphic in $\mathbb{C}$ and analytic on the unit circle. Suppose that for some $0 \leqslant \mu \leqslant 1$ and $1 \leqslant p \leqslant \infty$,

$$\limsup_{m \to \infty} e_f(\mu m; m; m; p)^{1/m \log m} \leqslant e^{\mu - 2/\rho}.$$

Then f is meromorphic of order $\leqslant \rho$.

5. Extensions

In this section, we mention without proof two theorems which extend, or relate to, the results in Sections 3 and 4. First, a result for functions with finite radius of analyticity.

THEOREM 5.1. (i) *Let $r > 1$. Let $f(z)$ be analytic in $|z| < r$, but not analytic in $|z| < r'$ for any $r' > r$. Then for $1 \leqslant p \leqslant \infty$ and $0 \leqslant \mu < \infty$,*

$$\limsup_{m \to \infty} e_f(\mu m/\log m; m; 0; p)^{1/m} = e^{\mu}/r. \tag{5.1}$$

(ii) *Assume that $f(z)$ is analytic in $|z| < 1$ and that (5.1) holds for some $1 \leqslant p \leqslant \infty, 0 \leqslant \mu < \infty$ and $r > 1$. Then f is the restriction to $|z| < 1$ of a function f analytic in $|z| < r$, but not in $|z| < r'$ for any $r' > r$.*

Next, we mention an extension of Theorem 3.2 whose proof may be based upon Theorem 3.2. It is in the same spirit as Theorems 1, 2 in Edrei [2] and Theorem 1 in Cirka [1]. Let us say that a function f has singularities of order $\rho < \infty$ in $\mathbb{C}$ if it has a representation

$$f(z) = h_0(z) \prod_{j=1}^{\nu} h_j(1/(z - z_j))$$

where v is a nonnegative integer, $z_1, z_2, \ldots, z_v \in \mathbb{C}$ and $h_0, h_1, \ldots, h_v$ are meromorphic of order $\rho_0, \rho_1, \ldots, \rho_v$, respectively, with $\rho = \sum_{j=0}^{v} \rho_j$.

THEOREM 5.2. *Let f have singularities of order at most $\rho < \infty$ in $\mathbb{C}$ and let f be analytic on the unit circle. Then for $0 \leqslant \mu < \infty$ and $1 \leqslant p \leqslant \infty$,*

$$\limsup_{m \to \infty} e_f(\mu m; m; m; p)^{1/m \log m} \leqslant e^{\mu - 1/p}.$$

6. THE ROLE OF THE LAST DERIVATIVE

In the previous sections, we have seen the significant role of the "last" derivative, for most of our estimates were based on the order of decrease of $\|(f-R)^{(\mu m)}\|$. We corroborate this role further here by proving that in some cases the best approximation of f in the norm $\|\cdot\|_{p,1,l}$ is equivalent to best approximation of $f^{(l)}$ in the norm $\|\cdot\|_{p,1}$. The proof is based upon the following lemma, which is a complex analogue of a result of Meir and Sharma [8, Theorem 3].

LEMMA 6.1. *Let $f(z)$ be analytic in $|z| < r$ with $\|f\|_{p,r,l} < \infty$ for some $1 \leqslant p \leqslant \infty$ and some positive integer l. Suppose that $f^{(j)}(0) = 0$, $j = 0, 1, 2, \ldots, l-1$. Then for $1 \leqslant p \leqslant \infty$, we have*

$$\|f^{(j)}\|_{p,r} \leqslant \frac{r^{l-j}}{(l-j)!} \|f^{(l)}\|_{p,r}, \qquad j = 0, 1, 2, \ldots, l-1 \tag{6.1}$$

and therefore

$$\|f^{(l)}\|_{p,r} \leqslant \|f\|_{p,r,l} \leqslant C \|f^{(l)}\|_{p,r}, \tag{6.2}$$

where $C = \max\{r^j/j! : j \geqslant 0\} = \max\{1, r^{[r]}/[r]!\}$, depends only on r. In particular for $r = 1$, we have

$$\|f\|_{p,1,l} = \|f^{(l)}\|_{p,1}. \tag{6.3}$$

Proof. The well-known Cauchy's formula for the solution of the initial value problem,

$$f^{(l)}(z) = g(z) \colon f(0) = f'(0) = \cdots = f^{(l-1)}(0) = 0$$

yields

$$|f^{(j)}(se^{i\theta})| = \left| \int_0^s \frac{(s-t)^{l-j-1}}{(l-j-1)!} f^{(l)}(te^{i\theta}) \, dt \right|$$

$j=0, 1, 2, ..., l-1$, $s<r$, $\theta\in[0, 2\pi]$. This equation may also be proved by integrating by parts. If we apply Minkowski's inequality for functions of two variables, namely

$$\left\{\int\left|\int Q(h,k)\,dk\right|^p dh\right\}^{1/p}\leqslant\int\left\{\int\left|Q(h,k)\right|^p dh\right\}^{1/p} dk,$$

to estimate $\|f^{(j)}\|_{p,s}$, $1\leqslant p<\infty$, we obtain

$$\begin{aligned}\|f^{(j)}\|_{p,s}&=\frac{1}{(l-j-1)!}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^s (s-t)^{l-j-1} f^{(l)}(te^{i\theta})\,dt\right|^p d\theta\right\}^{1/p}\\&\leqslant\frac{1}{(l-j-1)!}\int_0^s\left\{\frac{1}{2\pi}\int_0^{2\pi}|(s-t)^{l-j-1} f^{(l)}(te^{i\theta})|^p\,d\theta\right\}^{1/p} dt\\&\leqslant\frac{1}{(l-j-1)!}\int_0^s (s-t)^{l-j-1}\,\|f^{(l)}\|_{p,s}\,dt=\frac{s^{l-j}}{(l-j!)}\,\|f^{(l)}\|_{p,s}.\end{aligned}$$

The last inequality holds for any $s<r$ and this yields (6.1) for $1\leqslant p<\infty$. For $p=\infty$, the proof is similar and easier. Finally, (6.2) and (6.3) follow immediately from (6.1). ▌

We can now prove our equivalence result.

THEOREM 6.2. *Let $f(z)$ be analytic in $|z|<1$. Let m and l be positive integers such that $m>l$. Let $1\leqslant p\leqslant\infty$ and $\|f\|_{p,1,l}<\infty$. Let Q^* be the polynomial of best approximation of degree at most $m-l$ to $f^{(l)}$ in the norm $\|\cdot\|_{p,1}$, that is,*

$$\|f^{(l)}-Q^*\|_{p,1}=\min\{\|f^{(l)}-P\|_{p,1}: P\in\mathscr{P}_{m-l}\}.$$

Let P^* be the polynomial of degree at most m determined by the following conditions:

$$P^{*(j)}(0)=f^{*(j)}(0),\qquad j=0, 1, 2, ..., l-1,$$
$$P^{*(l)}=Q^*.$$

Then P^* is a polynomial of best approximation of degree at most m to f in the norm $\|\cdot\|_{p,1,l}$, that is

$$\|f-P^*\|_{p,1,l}=e_f(l; m; 0; p)=\|f^{(l)}-Q^*\|_{p,1}.$$

Proof. Let P be any polynomial of degree at most m. Then

$$\begin{aligned}\|f-P\|_{p,1,l} &\geqslant \|f^{(l)}-P^{(l)}\|_{p,1}\\ &\geqslant \|f^{(l)}-Q^*\|_{p,1}\\ &= \|f^{(l)}-P^{*(l)}\|_{p,1}\\ &= \|f-P^*\|_{p,1,l},\end{aligned}$$

by Lemma 6.1, as $(f-P^*)^{(j)}(0)=0$, $j=0, 1,..., l-1$. ∎

Remarks. (a) One can consider the following approximation problem:

$$\min_{P\in\mathscr{P}_n}\ \max_{0\leqslant j\leqslant l}\ \|f^{(k_j)}-P^{(k_j)}\|_{p,r},$$

where $0=k_0<k_1<\cdots<k_l\leqslant n$ (Lorentz [7]). Using (6.3) and Theorem 3 in [7], one can show that the polynomial of best approximation for the above problem is non-unique if $f\notin\mathscr{P}_n$ and $r^{k_1}/k_1!\leqslant 1$. In particular, this is always the case if $r\leqslant 1$.

(b) Proceeding in the same way as above one can show that

$$e_f(l; m; n; p)=\inf_{R\in\mathscr{R}_{mn}}\ \|f^{(l)}-R^{(l)}\|_{p,1}$$

provided $m-n\geqslant l$ and provided the rational functions R are restricted to have their poles in $\{z: |z|>1\}$. In particular under these restrictions, we have for $\mu<1$, ν a fixed positive integer and $m\geqslant\nu/(1-\mu)$,

$$e_f(\mu m; m; \nu; p)=\inf_{R\in\mathscr{R}_{m\nu}}\ \|f^{(\mu m)}-R^{(\mu m)}\|_{p,1}.$$

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